2-local isometies on spaces of continuous functions

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Abstract

We investigate the isometry groups of Banach algebras from the point of view of how they are determined by their local actions.

1 Introduction

Let $\mathcal{X}$ be a non-empty set. Let $\mathcal{M}(\mathcal{X})$ be the set of all maps from $\mathcal{X}$ into itself. Suppose that $\emptyset \neq \mathcal{S} \subset \mathcal{M}(\mathcal{X})$.

Definition 1. We say that $T \in \mathcal{M}(\mathcal{X})$ is 2-local in $\mathcal{S}$ if for every pair $x, y \in \mathcal{X}$ there exists $T_{x,y} \in \mathcal{S}$ such that

$$T(x) = T_{x,y}(x), \quad T(y) = T_{x,y}(y).$$

Definition 2. If every 2-local map in S is in fact an element of S, we say that S is 2-local reflexive in $\mathcal{M}(\mathcal{X})$.

Problem 3. *When is S 2-local reflexive in $\mathcal{M}(\mathcal{X})$?*

Motivated by an interesting extension by Kowalski and Słodkowski of the Gleason-Kahane-Żelazko theorem, Šemrl [15] initiated to study 2-local automorphisms and derivations. Probably besides the groups of the automorphisms and the derivations, most important class of transformations on a Banach algebra is the isometry group which reflects the geometrical properties of the underlying algebra. This motivates us to study the local properties of this group. Molnár [12] studied 2-local *complex-linear* surjective isometries of some operator algebras. After Molnár 2-local *complex-linear* surjective isometries on several spaces of continuous functions are studied by many authors [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12].

Molnár [13] mentioned the problem whether the group of all surjective isometires is 2-local reflexive or not. Although Molnár [14] has already proved among several interesting results that the group of all surjective isometries on $B(H)$ for a separable

Hilbert space is 2-local reflexive, the problem for $C(X)$ for a first countable compact Hausdorff space X, in particular $C([0,1])$, seems to be difficult. This problem of Molnár is much harder than that for the group of all surjective complex-linear isometries because of the fact that the number of the parameters is relatively large. In fact, If $U : C[0,1] \to C[0,1]$ is a surjective isometry, then

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in C[0,1],$$

$$U(f) = U(0) + \alpha \overline{f \circ \varphi}, \quad f \in C[0,1].$$

Hence the number of the parameters describing a surjective isometry on $C[0,1]$ is four, while the number of parameters for a surjective complex-linear isometry is two.

2 2-local reflexivity of $\mathrm{Iso}(C^1([0,1]))$

We study 2-local sujective isometries on the Banach algebra of complex-valued continuously differentiable functions $C^1[0,1]$ on the closed interval $[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0,1]$. The group of all surjective isometries on $C^1[0,1]$ is denoted by $\mathrm{Iso}(C^1[0,1])$. The representation theorem for $\mathrm{Iso}(C^1([0,1])$ is proved by Miura and Takagi [10].

Theorem 4 (Miura and Takagi)**.** *Let $U : C^1[0,1] \to C^1[0,1]$ be a surjective isometry. Then there exists a constant α of modulus 1 such that one of the following holds.*

(1) $U(f)(t) = U(0)(t) + \alpha f(t), \qquad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(2) $U(f)(t) = U(0)(t) + \alpha f(1-t), \quad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(3) $U(f)(t) = U(0)(t) + \alpha \overline{f(t)}, \quad \forall f \in C^1[0,1],\ \forall t \in [0,1],$
(4) $U(f)(t) = U(0)(t) + \alpha \overline{f(1-t)}, \quad \forall f \in C^1[0,1],\ \forall t \in [0,1].$

Theorem 5 ([5])**.** *The group $\mathrm{Iso}(C^1[0,1])$ is 2-local reflexive in $M(C^1[0,1])$.*

The above theorem states the following. Suppose that $T : C^1[0,1] \to C^1[0,1]$ is 2-local in $\mathrm{Iso}(C^1[0,1])$: i.e.,
$\forall f, g \in C^1[0,1]$, $\exists T_{f,g} \in \mathrm{Iso}(C^1[0,1])$ such that

$$T(f) = T_{f,g}(f), \qquad T(g) = T_{f,g}(g).$$

Then $T \in \mathrm{Iso}(C^1[0,1])$. Since $T_0 = T - T(0)$ is 2-local in $\mathrm{Iso}(C^1[0,1])$, we have by Lemma that

$\forall f, g \in C^1[0,1]$, $\exists \lambda_{f,g} \in C^1[0,1]$ and $\alpha_{f,g} \in \mathbb{C}$ of modulus 1 such that

$$T_0(f) = \lambda_{f,g} + \alpha_{f,g}(f \circ \varphi)^{\varepsilon_{f,g}} \text{ and } T_0(g) = \lambda_{f,g} + \alpha_{f,g}(g \circ \varphi)^{\varepsilon_{f,g}},$$

where $\varphi : [0,1] \to [0,1]$ is $\varphi = \mathrm{Id}$ or $1 - \mathrm{Id}$, and $(F)^{\varepsilon_{f,g}} = F$ or $\bar{F}$ depending on f and g. Note that the number of the parameters for T_0 is four. We show that

T_0 is a real-linear surjective isometry on $C^1[0,1]$. For every $c \in \mathbb{C}$, there exists $T_{c,0} \in \mathrm{Iso}(C^1[0,1])$ such that

$$\begin{aligned} T_0(c) = T_{c,0}(c) = \lambda_{c,0} + \alpha_{c,0}[c]^{\varepsilon_{c,0}} \\ 0 = T_0(0) = T_{c,0}(0) = \lambda_{c,0} + \alpha_{c,0}0 = \lambda_{c,0}. \end{aligned}$$

Thus $T_0(\mathbb{C}) \subset \mathbb{C}$:

Lemma 6. *$T_0(\mathbb{C}) \subset \mathbb{C}$, and $T_0|_{\mathbb{C}}$ is a real-linear isometry on $\mathbb{C}$.*

Hence there exists a complex number α of modulus 1 such that

$$T_0(z) = \alpha z \ (z \in \mathbb{C}) \text{ or } T_0(z) = \alpha \bar{z} \ (z \in \mathbb{C}).$$

The point is to consider the set

$$W = \{f \in C^1[0,1] : \text{If } U(f([0,1])) = f([0,1]) \text{ for an isometry on } \mathbb{C}, \text{then } U \text{ is the identity}\}.$$

Note that : $U(z) = \lambda + \alpha z \ (z \in \mathbb{C})$ or $U(z) = \lambda + \alpha \bar{z} \ (z \in \mathbb{C})$. Let P be the set of all polynomials. Many polynomials are in W:

- $t + it^2$
- ...
- ...

But it is not always the case:

- $(t-1/2)^3 + i(t-1/2)^2$

Lemma 7. *$P \subset \overline{W}$, the uniform closure of W. Hence W is uniformly dense in $C^1[0,1]$.*

Let

$$w(t) = \begin{cases} 0, & t = 0 \\ t^3 \sin \frac{1}{t}, & 0 < t \leq 1 \end{cases}$$

For $f = p + iq \in P$ and $m \in \mathbb{N}$, put

$$f_m = \begin{cases} iw(\frac{1}{m} - t) + \left(p'\left(\frac{1}{m}\right) + iq'\left(\frac{1}{m}\right)\right)\left(t - \frac{1}{m}\right) + p\left(\frac{1}{m}\right) + iq\left(\frac{1}{m}\right), & 0 \leq t \leq \frac{1}{m} \\ p(t) + iq(t), & \frac{1}{m} \leq t \leq 1 \end{cases}$$

Then

$$\{f_m : f = p + iq \in W, p \text{ is not constant and } p,\, q,\, 1 \text{ is linearly independent}\} \subset W.$$

Lemma 8. *Suppose that $T_0(z) = \alpha z$ $(z \in \mathbb{C})$. Then*

$$T_0(f)(t) = \alpha f(t) \text{ or } T_0(f)(t) = \alpha f(1-t) \text{ for } f \in W.$$

Suppose that $T_0(z) = \alpha \bar{z}$ $(z \in \mathbb{C})$. Then

$$T_0(f)(t) = \alpha \overline{f(t)} \text{ or } T_0(f)(t) = \alpha \overline{f(1-t)} \text{ for } f \in W.$$

We show how to use W to reduce the number of the parameters for the case where $T_0(z) = z$ $(z \in \mathbb{C})$.

Let $f \in W$. By the property of 2-localness for f and 0 we have

$$T_0(f) = \lambda_{f,0} + \alpha_{f,0}(f \circ \varphi_{f,0})^{\varepsilon_{f,0}}, \qquad 0 = T_0(0) = \lambda_{f,0} + \alpha_{f,0}0.$$

Then $\lambda_{f,0} = 0$ follows and we have

$$T_0(f) = \alpha_{f,0}(f \circ \varphi_{f,0})^{\varepsilon_{f,0}}.$$

Let $0 \neq c \in \mathbb{C}$ be arbitrary and fix it. We also have that

$$T_0(f) = \lambda_{f,c} + \alpha_{f,c}(f \circ \varphi_{f,c})^{\varepsilon_{f,c}}, \quad c = T_0(c) = \lambda_{f,c} + \alpha_{f,c}(c)^{\varepsilon_{f,c}}.$$

By the second equation, $\lambda_{f,c}$ is a constant. Then

$$\alpha_{f,0}(f \circ \varphi_{f,0})^{\varepsilon_{f,0}} = \lambda_{f,c} + \alpha_{f,c}(f \circ \varphi_{f,c})^{\varepsilon_{f,c}}.$$

From

$$\alpha_{f,0}(f \circ \varphi_{f,0})^{\varepsilon_{f,0}} = \lambda_{f,c} + \alpha_{f,c}(f \circ \varphi_{f,c})^{\varepsilon_{f,c}}$$

we have four possibility depending on $\varepsilon_{f,0}$ and $\varepsilon_{f,c}$.

(1) $f \circ \varphi_{f,0} = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c} f \circ \varphi_{f,c}$,
(2) $f \circ \varphi_{f,0} = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}\overline{f \circ \varphi_{f,c}}$,
(3) $f \circ \varphi_{f,0} = \alpha_{f,c}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}} f \circ \varphi_{f,c}$,
(4) $f \circ \varphi_{f,0} = \alpha_{f,c}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}}\overline{f \circ \varphi_{f,c}}$.

Considering the range of these equations we have

(1) $f([0,1]) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c} f([0,1])$,
(2) $f([0,1]) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}\overline{f([0,1])}$,
(3) $f([0,1]) = \alpha_{f,c}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}} f([0,1])$,
(4) $f([0,1]) = \alpha_{f,c}\overline{\lambda_{f,c}} + \alpha_{f,0}\overline{\alpha_{f,c}}\overline{f([0,1])}$.

Since $f \in W$, (2) and (4) are impossible. In fact, letting an isometry $S(z) = \overline{\alpha_{f,0}}\lambda_{f,c} + \overline{\alpha_{f,0}}\alpha_{f,c}\bar{z}$ $(z \in \mathbb{C})$, (2) means that

$$f([0,1]) = S(f([0,1])),$$

which is impossible for S being not the identity. Hence (2) is impossible. (4) is impossible in the same way.

We also see that (3) is impossible by some different reason. This is a part of the proof applying the property of W. By a further consideration we see that $T_0(f) = f \circ \varphi_{f,0}$ when $T_0(z) = z$ $(z \in \mathbb{C})$. We need to prove that $\varphi_{f,0}$ does not depend on f. To prove it we first prove that $T_0(Id) = Id$ or $T_0(Id) = 1 - Id$. This can be proved by an aproximation argument. If $T_0(z) = \alpha z$ $(z \in \mathbb{C})$ and $T_0(Id) = Id$, then

$$T_0(f)(t) = \alpha f(t), \quad \forall f \in W.$$

If $T_0(z) = \alpha z$ ($z \in \mathbb{C}$) and $T_0(Id) = 1 - Id$, then

$$T_0(f)(t) = \alpha f(1-t), \quad \forall f \in W.$$

If $T_0(z) = \alpha \bar{z}$ ($z \in \mathbb{C}$) and $T_0(Id) = Id$, then

$$T_0(f)(t) = \alpha \overline{f(t)}, \quad \forall f \in W.$$

If $T_0(z) = \alpha \bar{z}$ ($z \in \mathbb{C}$) and $T_0(Id) = 1 - Id$, then

$$T_0(f)(t) = \alpha \overline{f(1-t)}, \quad \forall f \in W.$$

As W is uniformly dense in $C^1[0,1]$ we conclude that:

If $T_0(z) = \alpha z$ ($z \in \mathbb{C}$) and $T_0(Id) = Id$, then

$$T_0(f)(t) = \alpha f(t), \quad \forall f \in C^1[0,1].$$

If $T_0(z) = \alpha z$ ($z \in \mathbb{C}$) and $T_0(Id) = 1 - Id$, then

$$T_0(f)(t) = \alpha f(1-t), \quad \forall f \in C^1[0,1].$$

If $T_0(z) = \alpha \bar{z}$ ($z \in \mathbb{C}$) and $T_0(Id) = Id$, then

$$T_0(f)(t) = \alpha \overline{f(t)}, \quad \forall f \in C^1[0,1].$$

If $T_0(z) = \alpha \bar{z}$ ($z \in \mathbb{C}$) and $T_0(Id) = 1 - Id$, then

$$T_0(f)(t) = \alpha \overline{f(1-t)}, \quad \forall f \in C^1[0,1].$$

3 2-local reflexivity of $\mathrm{Iso}(\mathrm{Lip}(K))$

For a compact metric space K, let

$$\mathrm{Lip}(K) = \left\{ f \in C(K) : L_f = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty \right\}$$

with the norm $\|f\|_\Sigma = \|f\|_\infty + L_f$ for $f \in \mathrm{lip}_\alpha(K)$. We say that L_f is the Lipschitz constant for f. With this norm $\mathrm{lip}_\alpha(K)$ is a unital semisimple commutative Banach algebra. We prove the following in [5].

Theorem 9 ([5])**.** *Let K_j be a compact metric space for $j = 1, 2$. Suppose that $U : \mathrm{lip}_\alpha(K_1) \to \mathrm{lip}_\alpha(K_2)$ is a surjective real-linear isometry with respect to the norm $\|f\|_\Sigma = \|f\|_\infty + L_f$ for $f \in \mathrm{lip}_\alpha(K_1)$. Then there exists a surjective isometry $\pi : K_2 \to K_1$ such that*

$$U(f) = U(1) f \circ \pi, \qquad f \in \mathrm{lip}_\alpha(K_1)$$

or

$$U(f) = U(1) \overline{f \circ \pi}, \qquad f \in \mathrm{lip}_\alpha(K_1).$$

Applying Theorem 9, in the similar way as in Section 2 we see the following.

Theorem 10 ([5])**.** $\mathrm{Iso}(\mathrm{Lip}[0,1])$ *is 2-local reflexive in* $M(\mathrm{Lip}[0,1])$.

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